a = ?

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Let $n \geq 2$ be an integer.

Find all real numbers a such that there exist real numbers $x_1, ..., x_n$ satisfying $x_1(1-x_2) = x_2(1-x_3) = ... = x_{n-1}(1-x_n) = x_n(1-x_1) = a$.

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Let A be set all real numbers a such that system of equations

(1) $\begin{cases} x_k(1-x_{k+1}) = a, k = 1,2,..,n-1 \\ x_n(1-x_1) = a \end{cases}$

is solvable with respect to $x_1, ..., x_n \in \mathbb{R}$.

Noting that for $a = 0$ the system **(1)** has obvious solution $x_1 = x_2 = ... = x_n = 0$ we assume further that $a \neq 0$.This immediately implies that $x_i \neq 0, i = 1,2,...,n$ and we can rewrite the system as follows:

(2) $\begin{cases} x_{k+1} = h(x_k), k = 1,2,..,n-1 \\ x_1 = h(x_n) \end{cases}$, where $h(x) := 1 - \dfrac{a}{x} = \dfrac{x-a}{x}$.

Let $h_1(x) := h(x), h_{n+1}(x) = h(h_n(x)), n \in \mathbb{N}$ and H_n be matrix of coefficients for

Mobius function $h_n(x)$,that is $h_n(x) = \dfrac{a_n x + b_n}{c_n x + d_n}$ and $H_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, n \in \mathbb{N}$.

Also let $h_0(x) := x$. Then $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H_1 = H = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$ and $H_{n+1} = H \cdot H_n \iff$

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} =$$

$$\begin{pmatrix} a_n - ac_n & b_n - ad_n \\ a_n & b_n \end{pmatrix} \iff \begin{cases} a_{n+1} = a_n - ac_n \\ b_{n+1} = b_n - ad_n \\ c_{n+1} = a_n \\ d_{n+1} = b_n \end{cases} \iff$$

$$\begin{cases} a_{n+1} = a_n - aa_{n-1} \\ b_{n+1} = b_n - ab_{n-1} \\ c_{n+1} = a_n \\ d_{n+1} = b_n \end{cases}, n \in \mathbb{N} \text{ and } a_0 = 1, a_1 = 1, b_0 = 0, b_1 = -a.$$

Since (a_n) and (b_n) satisfies to the same recurrence and $b_2 = -a$ then $b_n = -aa_{n-1}, n \in \mathbb{N}$.

Thus, $H_n = \begin{pmatrix} a_n & -aa_{n-1} \\ a_{n-1} & -aa_{n-2} \end{pmatrix}, n \geq 2$ and $h_n(x) = \dfrac{a_n x - aa_{n-1}}{a_{n-1}x - aa_{n-2}}, n \geq 2$.

Coming back to the system **(2)** we can see that $x_k = h_k(x_1), k = 1,2,..,n-1$ and $x_1 = h_n(x_1)$,

that is x_1 is solution of equation $h_n(x) = x$.Thus $A_n = \{a \mid h_n(x) = x, x \in \mathbb{R}\}$.

Since $h_n(x) = x \iff \dfrac{a_n x - aa_{n-1}}{a_{n-1}x - aa_{n-2}} = x \iff a_n x - aa_{n-1} = a_{n-1}x^2 - aa_{n-2}x \iff$

(3) $a_{n-1}x^2 - x(a_n + aa_{n-2}) + aa_{n-1} = 0$,

where a_n is polynomial of a defined recursively by

$a_{n+1} = a_n - a a_{n-1}, n \in \mathbb{N}$, $a_0 = 1, a_1 = 1$

and quadratic equation **(3)** is solvable in real x iff its discriminant

$D_n := (a_n + a a_{n-2})^2 - 4 a a_{n-1}^2 = a^2 a_{n-2}^2 + 2 a a_n a_{n-2} - 4 a a_{n-1}^2 + a_n^2 =$

$a^2 a_{n-2}^2 - 4 a a_{n-1}^2 + 2 a a_{n-2} (a_{n-1} - a a_{n-2}) + (a_{n-1} - a a_{n-2})^2 = a_{n-1}^2 (1 - 4a) =$
$a_{n-1}^2 (1 - 4a)$ is

non negative then

$A_n = \left\{ a \mid a_{n-1}^2 (1 - 4a) \geq 0 \right\} = (-\infty, 1/4] \cup \{ a \mid a_{n-1} = 0 \}, n \geq 2.$

For example, $a_2 = 1 - a, a_3 = 1 - 2a, , a_4 = a^2 - 3a + 1, a_5 = a^2 - 3a + 1 - a(1 - 2a) =$

$3a^2 - 4a + 1$ and $A_2 = (-\infty, 1/4]$, $A_3 = (-\infty, 1/4] \cup \{1\}, A_4 = (-\infty, 1/4] \cup \{1/2\},$

$A_5 = (-\infty, 1/4] \cup \left\{ \dfrac{3 - \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2} \right\}.$

Note that for any $a \leq \dfrac{1}{4}$ system **(1)** solvable in $\mathbb{R}$. Indeed, since

$$h(x) = x \iff x^2 - x + a = 0 \iff x \in \left\{ \frac{1 - \sqrt{1 - 4a}}{2}, \frac{1 + \sqrt{1 - 4a}}{2} \right\}$$

then $(x_1, x_2, ..., x_n) = (x, x, x, ..., x)$ for any such x is solution of **(1)** because for $x_1 = x$ we have

$$h_k(x_1) = h_k(x) = x, k = 1, 2, ..., n.$$

Therefore, to complete the solution of the problem remains find all solution of equation

$a_{n-1}(a) = 0$ in real $a > 1/4$ for any $n \geq 2$.

Since $a > 1/4 \iff \dfrac{1}{2\sqrt{a}} < 1$ then denoting $\alpha := \arccos \dfrac{1}{2\sqrt{a}}$ and $b_n := \dfrac{a_n}{(\sqrt{a})^n}$ we obtain

$$a_{n+1} = a_n - a a_{n-1} \iff \frac{a_{n+1}}{(\sqrt{a})^{n+1}} - \frac{1}{\sqrt{a}} \cdot \frac{a_n}{(\sqrt{a})^n} + \frac{a_{n-1}}{(\sqrt{a})^{n-1}} = 0 \iff$$

(4) $\quad b_{n+1} - 2\cos\alpha \cdot b_n + b_{n-1} = 0, n \in \mathbb{N}.$

Since $b_n = c_1 \cos n\alpha + c_2 \sin n\alpha$ and $b_0 = 1, b_1 = \dfrac{1}{\sqrt{a}} = 2\cos\alpha$ we obtain $c_1 = 1, c_2 = \cot\alpha$

and, therefore, $b_n = \cos n\alpha + \cot\alpha \sin n\alpha = \dfrac{\sin(n+1)\alpha}{\sin\alpha}, n \in \mathbb{N}.$

Thus, for any $n \geq 2$ we have $a_n = \dfrac{a^{n/2}\sin(n+1)\alpha}{\sin\alpha}$ and $a_n = 0 \iff$

$$\begin{cases} \sin(n+1)\alpha = 0 \\ \sin\alpha \neq 0 \\ a = \dfrac{1}{4\cos^2\alpha} \end{cases} \iff$$

$$\begin{cases} \alpha = \dfrac{1}{4\cos^2 \dfrac{k\pi}{n+1}} \\ k = 1, 2, ..., n \\ a = \dfrac{1}{4\cos^2 \alpha} \end{cases} \iff a = \dfrac{1}{4\cos^2 \dfrac{k\pi}{n+1}}, k = 1, 2, ..., \left\lfloor \dfrac{n+1}{2} \right\rfloor$$

(since $\cos^2 \dfrac{k\pi}{n+1} = \dfrac{(n+1-k)\pi}{n+1}$, $k = 1, 2, .., n$).
Thus, for any $n \geq 2$ equation $h_n(x) = x$ solvable in $\mathbb{R}$ Iff

$$a \in A_n = (-\infty, 1/4] \cup \left\{ \dfrac{1}{4\cos^2 \dfrac{k\pi}{n}} \mid k = 1, 2, ..., \left\lfloor \dfrac{n}{2} \right\rfloor \right\}$$